A Brief Introduction to Ray Tracing and Ionospheric Models

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- The formulation is generalized in that it is a way to describe propagation through many media.

Important:

You are only as accurate as the ionospheric model you use in the ray tracer!

Talk Organization

- Where does the ray tracer come from? What are you attempting to solve?
  - Index of refraction: Appleton-Hartree equation
  - Snell's Law
  - Hamilitonian Optics formulation, aka, the Haselgrove equations
- What are the different options out there for ray tracers and ionospheric models?

Appleton Hartree Index of refraction - I

The Appleton Hartree equation describes the propagation of an electromagnetic wave in a magnetized plasma.

Starting with Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial B}{\partial t}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

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Using a plane-wave approximation that, $A \propto \tilde{A} \exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t)$[$i$ is the imaginary number - physics convention], then derivatives become

$$\frac{\partial}{\partial t} \rightarrow -i\omega$$
$$\frac{\partial}{\partial \mathbf{x}} \rightarrow i\mathbf{k}$$

Appleton Hartree Index of refraction - II

We take the two equations from Maxwell's equation, take the curl of Ampere's law to arrive at the following relation:

$$-\nabla \times \nabla \times \mathbf{E} = \mu_0 \frac{\partial \mathbf{j}}{\partial t} + \epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

which becomes,

$$\mathbf{k} \times \mathbf{k} \times \tilde{\mathbf{E}} = \mu_0 i\omega \tilde{\mathbf{j}} + \frac{\omega^2}{c^2} \tilde{\mathbf{E}}$$

And that $\mathbf{j} = \sigma \cdot \mathbf{E}$ and that

$$\mathbf{K} = \mathbf{1} - \frac{\sigma}{i\omega\epsilon_0}$$

we arrive at the key relation:

$$\mathbf{k} \times (\mathbf{k} \times \tilde{\mathbf{E}}) + \tilde{\mathbf{K}} \cdot \tilde{\mathbf{E}} = 0 \tag{1}$$

Appleton Hartree Index of refraction - III

Now lets evaluate $\mathbf{K}$, our goal is to arrive at a relation that is in the form $\mathbf{j} = n_e e \mathbf{U_e}$, we will start with the Lorentz force for a particle in a magnetic field, where $\mathbf{B} = B_0 \hat{z}$ is in the z direction in an x,y,z coordinate system.

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$$m_e \frac{d\mathbf{U}_e}{dt} = q_e(\mathbf{E} + \mathbf{U} \times \mathbf{B})$$

which in terms of components becomes and applying our relations for derivatives,

$$-i\omega m_e \tilde{U_{1x}} = q_e \left(\tilde{E_{1x}} + \tilde{U_{1y}} B_0 \right)$$
$$-i\omega m_e \tilde{U_{1y}} = q_e \left(\tilde{E_{1y}} - \tilde{U_{1x}} B_0 \right)$$
$$-i\omega m_e \tilde{U_{1z}} = q_e \tilde{E_{1z}}$$

Appleton Hartree Index of refraction - III

We can re-write into a matrix equation

$$\begin{bmatrix} -i\omega & -\Omega_e & 0 \\ \Omega_e & -i\omega & 0 \\ 0 & 0 & -i\omega \end{bmatrix} \begin{bmatrix} \tilde{U_{1x}} \\ \tilde{U_{1y}} \\ \tilde{U_{1z}} \end{bmatrix} = \begin{bmatrix} \tilde{E_{1x}} \\ \tilde{E_{1y}} \\ \tilde{E_{1z}} \end{bmatrix}$$

We can then invert this matrix and pressing forward with a few step to get it into $\tilde{j} = n_e q_e \mathbf{\tilde{U}_e} = \sigma \cdot \mathbf{\tilde{E}}$,

$$\frac{\sigma}{i\omega\epsilon_0} = \frac{\omega_{pe}^2}{i\omega(\Omega_e^2 - \omega^2)} \begin{bmatrix} -i\omega & \Omega_e & 0 \\ -\Omega_e & -i\omega & 0 \\ 0 & 0 & -i/\omega \end{bmatrix} \quad (2)$$

where $\omega_{pe}^2 = \frac{n_e q_e^2}{\epsilon_0 m_e}$ and $\Omega_e = \frac{q_e B_0}{m_e}$

Appleton Hartree Index of refraction - IV

Combining the Equation 1 and 2 (with a little more work), and using

$$\mathbf{n} = \frac{\mathbf{k}c}{\omega} = [n\sin\theta, 0, n\cos\theta]$$

$$\mathbf{D}(\mathbf{k},\omega)\cdot\mathbf{E} = \begin{bmatrix} S - n^2\cos^2\theta & -iD & n^2\sin\theta\cos\theta \\ iD & S - n^2 & 0 \\ n^2\sin\theta\cos\theta & 0 & P - n^2\sin\theta^2 \end{bmatrix} \begin{bmatrix} \tilde{E_{1x}} \\ \tilde{E_{1y}} \\ \tilde{E_{1z}} \end{bmatrix} = 0$$

$$S = 1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2}$$

$$D = \frac{\Omega_e\omega_{pe}^2}{\omega(\omega^2 - \Omega^2)}$$

$$P = 1 - \frac{\omega_{pe}^2}{\omega^2}$$

Appleton Hartree Index of refraction - V

Solving for $det[D(k,\omega)] = 0$ and a few steps later (in Jackson form), we get (neglecting collisions)

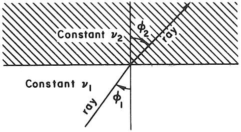
$$n^2 = 1 - \frac{X(1-X)}{1 - X - \frac{1}{2}Y^2\sin^2\theta \pm \left((1-X)^2 Y^2\cos^2\theta + (\frac{1}{2}Y^2\sin^2\theta)^2\right)^{1/2}}$$

$$X = \frac{\omega_{pe}^2}{\omega^2} = \frac{q_e^2 n_e(r,\theta,\phi)}{m_e\epsilon_0\omega}$$

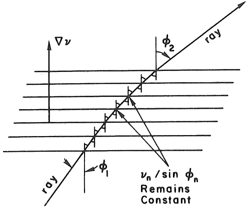
$$Y = \frac{\Omega_e}{\omega} = \frac{q_e B_0}{m_e\omega}$$

Where the O- and X-mode corresponds to the $+$ and $-$ in the AH equation, and corresponds to LH and RH circular polarization in the northern hemisphere, respectively. For $\theta = 90^\circ$, the $+$ and $-$ corresponds to the ordinary and extraordinary mode, respectively.

Snell's Law



a. A single application of Snell's law



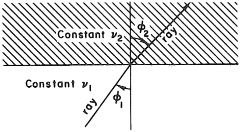
b. A repeated application

A simple way to ray trace would be to iteratively solve Snell's law assuming given that you can calculate the index of refraction, n.

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

where θ is defined relative to the normal of the layer.

Snell's Law



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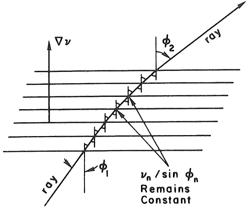

b. A repeated application

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$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

where θ is defined relative to the normal of the layer. Could also derive a differential form,

$$d(n \sin\theta) = constant \rightarrow$$
$$dn \sin\theta - n\cos\theta d\theta = 0 \rightarrow$$
$$\frac{d\theta}{dn} = \tan\theta$$

Snell's Law for Spherical Symmetry

Bouger's Rule

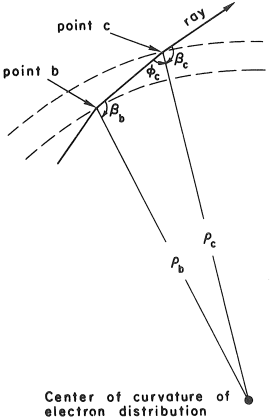
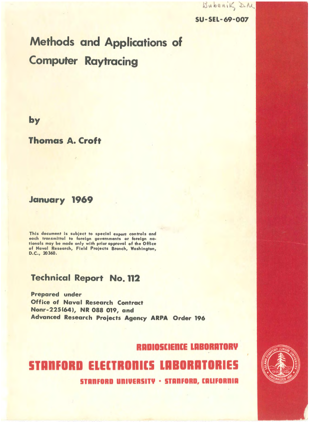


Fig. 27. THE GEOMETRY USED TO DERIVE BOUGER'S RULE.



Bubenik, D.M.

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Methods and Applications of
Computer Raytracing

by

Thomas A. Croft

January 1969

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RADIOSCIENCE LABORATORY

STANFORD ELECTRONICS LABORATORIES

STANFORD UNIVERSITY · STANFORD, CALIFORNIA

Stanford Mark X Ray Tracer.

$$n_b \rho_b \sin\beta_b = n_c \rho_c \sin\beta_c$$

Lagrangian and Hamilitionian Mechanics

A well-known method for solving equations of motions of mechanical systems are the Lagrangian and Hamilitionian mechanics formulation. The idea is based on the concept of finding the extrema of a quantity.

$$\delta \int L(q_j, \dot{q}_j, t)dt = 0$$

"Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval, the actual path followed is that which is minimizes the time integral of difference between the kinetic and potential energy" [Marion and Thorton, 2004]

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Lagrangian Formulation

$$\delta \int L(q_j, \dot{q}_j, t) dt = 0$$

$$L(q_j, \dot{q}_j, t) = T(\dot{q}_j) - U(q_j)$$

$$\frac{dL}{dq_j} - \frac{d}{dt}\frac{dL}{d\dot{q}_j} = 0$$

Hamilitionian Formulation:

$$H(q_j, p_j, t) = T(\dot{q}_j) + U(q_j)$$

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}$$

$$\frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j}$$

Hamiltionian Optics Formulation of Ray Tracing

Figure 1. Jenifer Haselgrove

Haselgrove 1954 derived a relative general formulation describing rays being traced through an anisotropic medium.
Starting from Fermat's principle

$$\delta \int \mu \cos \alpha ds = 0$$

Haselgrove derived the Hamilitionian:

$$\frac{dp}{dt} = \frac{du^i}{dt} = -g^{ij}\frac{dH}{dx_i} - g^{ij}g^{km}u^l\frac{\partial H}{\partial u^m}\left(\frac{\partial g_{li}}{\partial x_k} - \frac{\partial g_{kl}}{\partial x_i}\right)$$
$$\frac{dq}{dt} = \frac{dx_j}{dt} = g^{ij}\frac{\partial H}{\partial u^i}$$

Hamiltionian Optics Formulation of Ray Tracing

Algorithm was developed by Jones and Stephenson circa 1975 which numerically solved the Haselgrove equations. In effect, the algorithm these first order differential equations in spherical coordinates:

$$H = \frac{1}{2}\left(\frac{c^2}{\omega^2}(k_r^2 + k_\theta^2 + k_\phi^2) - n^2\right)$$

$$\frac{dr}{dt} = \frac{dH}{dk_r}$$
$$\frac{d\theta}{dt} = \frac{1}{r}\frac{dH}{dk_\theta}$$
$$\frac{d\phi}{dt} = \frac{1}{r\sin\theta}\frac{dH}{dk_\phi}$$
$$\frac{dt'}{dt} = \frac{-dH}{d\omega}$$

$$\frac{dk_r}{dt} = -\frac{dH}{dr} + k_\theta\frac{d\theta}{dt} + k_\phi \sin\theta\frac{d\phi}{dt}$$
$$\frac{dk_\theta}{dt} = \frac{1}{r}\left(-\frac{dH}{d\theta} - k_\theta\frac{dr}{dt} + k_\phi r\cos\theta\frac{d\phi}{dt}\right)$$
$$\frac{dk_\phi}{dt} = \frac{1}{r\sin\theta}\left(-\frac{dH}{d\phi} - k_\phi\sin\theta\frac{dr}{dt} + k_\phi r\cos\theta\frac{d\theta}{dt}\right)$$
$$\frac{d\omega}{dt} = \frac{\partial H}{\partial t'}$$

where t is an independent parameter which will become group path $(P' = ct')$, and $k_r^2 + k_\theta^2 + k_\phi^2 = \omega^2/c^2$

Hamiltionian Optics Formulation of Ray Tracing

Group Delay:

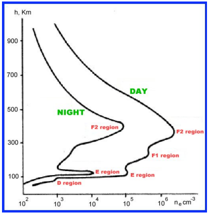
$$P' = ct'$$

Phase path (time changes in phase path are Doppler shifts):

$$\frac{dP}{dP'} = -\frac{1}{\omega}\frac{k_r \frac{\partial H}{\partial k_r} + k_\theta \frac{\partial H}{\partial k_\theta} + k_\phi \frac{\partial H}{\partial k_\phi}}{\frac{\partial H}{\partial \omega}}$$

Can also calculate other quantities like optical path length, absorption, and polarization.

Ionospheric Models



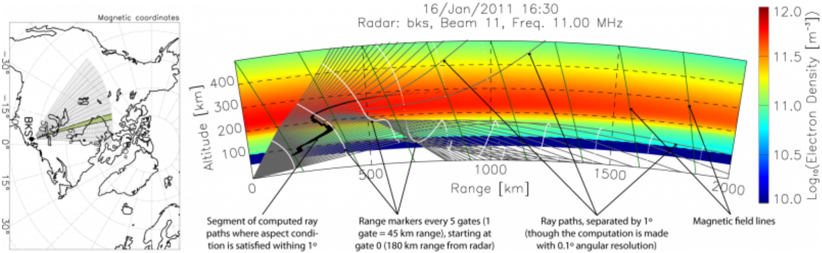
Important:

You are only as accurate as the ionospheric model you use in the ray tracer!

- Analytic Expressions: Parabolic, Quasi-parabolic, Chapman Layer (derived from theory of how ionization occurs), Bent
- International Reference Ionosphere - derived from ionosonde, ISR, and many other data sources, although does use analytic expressions and interpolation. State of the art model!
- Other "research grade" models not easily available: SAMI-3, IDA4D, etc.

Ray Tracers, Types, and Models Used

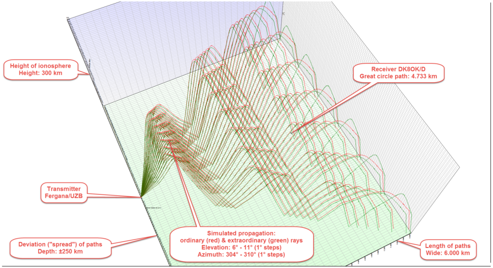
Superdarn Raydarn: 2-D ray tracer found in Davitpy



- 2-D Ray tracer without magnetic field. Version of the Coleman [1991 DTSO Document, Radio Science 1998] algorithm which uses a 2-D Lagrangian formulation.
- Advantages: Integrated with python and davitpy, parallelized, open source
- Disadvantages: only 2-D and does not include magnetic field effects

Ray Tracers, Types, and Models Used

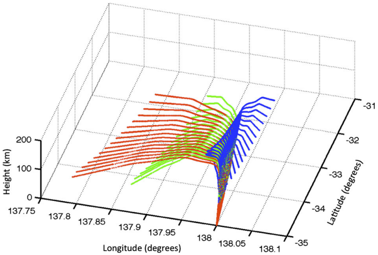
Proplab Pro



- At least a 2-D ray tracer (probably 3-D) within IRI including full magnetoionic effects (X and O mode).
- Advantage: Most user friendly including graphics, “vetted” in amateur community.
- Disadvantage: Closed source, $240(!) for the software

Ray Tracers, Types, and Models Used

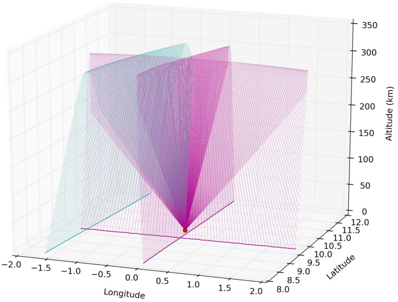
PHARLARP



- Developed by DTSO, 2-D and 3-D algorithms, including solving Haselgrove equations with full magnetoionic effects. Also includes other useful things, such as calculation of noise, etc.
- Advantage: Matlab interface, full bore "free" research-grade ray tracer
- Disadvantage: Matlab interface, semi-open source (some of the key code is precompiled), must ask for permission to use it[See Cuevera et al., 2014, Radio Science].

Ray Tracers, Types, and Models Used

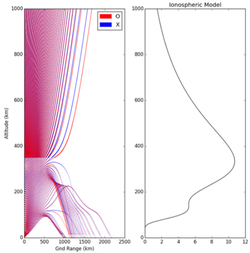
MoJo (Modified Jones-Stephenson)



- Developed by Kate Zawdie at NRL, basically modernized version of Jones-Stephenson algorithm.
- Advantage: Full 3-D Ray Tracer, has been coupled with SAMI-3 ionospheric model and other models at NRL
- Disadvantage: Closed source and possibly proprietary(?)

Ray Tracers, Types, and Models Used

Original Jones-Stephenson Algorithm



- The gold standard.
- Advantages: fast, modular, at least one other ray tracer basically uses this directly (IONORT - Italian group), source is online, GREAT Documentation!
- Disadvantages: written in Fortran 4 (!), ionospheric models need to be compiled with the code, use of common blocks, memory leaks.

Toward an Open Ray Tracer and Summary

- Hopefully have given some insight about where index of refraction and ray tracing equations come from and a terse overview of the theoretical underpinning.
- One strategic push forward is to develop a generalized, modernized, open source ray tracing software toolkit that could be both by researchers and hopefully amateurs (more work required).

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